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ON PERMANENT ROTATIONS OF A HEAVY SOLID BODY ON AN ABSOLUTELY ROUGH HORIZONTAL PLANE*

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Conditions of existence and stability of permanent rotations of a heavy solid body on an absolutely rough horizontal plane are obtained. A well-defined analogy of the investigated problem to problems of permanent rotations of a body on an absolutely smooth plane and of a body with a fixed point is pointed out, as well their essential differences. In the case of the absolutely rough plane an asymptotic stability of permanent rotations is possible with respect to a part of variables, although the system is conservative. Stability then depends on the direction of rotation.

1. Consider a heavy solid body bounded by a convex surface on an absolutely rough horizontal plane. Position of the body is defined by the coordinates x and y of its center of mass in the fixed system of coordinates $Oxyz$ (the plane Oxy coincides with the horizontal supporting plane and the Oz axis is directed vertically upward) and by Euler's angles θ, φ, ψ of the principal central axes $G\xi, G\eta, G\zeta$ of inertia of the body to the axes of the fixed coordinate system. Then the Lagrange function and the system constraint equations that express the absence of slipping at the point of contact of the body with the plane assume the form

$$\begin{aligned}
&L = {}^1/_2\, [A\cos^2\varphi + B\sin^2\varphi + m(\chi_1\cos\theta - \zeta\sin\theta)^2]\,\theta^{\cdot 2} + \\
&\quad {}^1/_2\,(C + m\chi_2^2\sin^2\theta)\,\varphi^{\cdot 2} + {}^1/_2\,[(A\sin^2\varphi + B\cos^2\varphi)\sin^2\theta + \\
&\quad C\cos^2\theta]\,\psi^{\cdot 2} + m(\chi_1\cos\theta - \zeta\sin\theta)\,\chi_2\sin\theta\,\theta^\cdot\varphi^\cdot + \\
&\quad (A-B)\sin\theta\sin\varphi\cos\varphi\,\theta^\cdot\psi^\cdot + C\cos\theta\,\varphi^\cdot\psi^\cdot + {}^1/_2\, m \times \\
&\quad (x^{\cdot 2} + y^{\cdot 2}) + mg(\chi_1\sin\theta + \zeta\cos\theta) \\
&x^\cdot = \alpha_1\theta^\cdot + \alpha_2\varphi^\cdot + \alpha_3\psi^\cdot,\quad y^\cdot = \beta_1\theta^\cdot + \beta_2\varphi^\cdot + \beta_3\psi^\cdot \\
&\alpha_1 = -(\chi_1\sin\theta + \zeta\cos\theta)\sin\psi,\ \alpha_2 = \chi_2\cos\theta\sin\psi + \chi_1\cos\psi \\
&\alpha_3 = \chi_2\sin\psi + (\chi_1\cos\theta - \zeta\sin\theta)\cos\psi,\ \beta_i = -\partial\alpha_i/\partial\psi \\
&(i = 1, 2, 3) \\
&\chi_1 = \xi\sin\varphi + \eta\cos\varphi,\ \chi_2 = \xi\cos\varphi - \eta\sin\varphi
\end{aligned}$$

where m is the mass of body, A, B, C its principal central moments of inertia, ξ, η, ζ are the coordinates of point K of contact of the body with the supporting plane in the coordinate system $G\xi\eta\zeta$. It can be shown that ξ, η, ζ are functions of variables θ and φ that are determined by the form of the equation defining the body surface and satisfying two relations of the form

$$(\xi'\sin\varphi + \eta'\cos\varphi)\sin\theta + \zeta'\cos\theta \equiv 0 \tag{1.1}$$

where the prime indicates differentiation with respect to θ or φ.

A heavy solid body on an absolutely rough horizontal plane, obviously, represents a nonholonomic Chaplygin system (nonintegrable constraints, and the Lagrange function and the constraint coefficients are independent of x and y). Motions of the body are defined by Chaplygin's equations which in this case are of the form

$$\frac{d}{dt}\frac{\partial L^*}{\partial q_i^\cdot} = \frac{\partial L^*}{\partial q_i} + \sum_{j,k=1}^{3}\Gamma_{ijk}q_j^\cdot q_k^\cdot \quad (i = 1, 2, 3;\ q_1 = \theta,\ q_2 = \varphi,\ q_3 = \psi) \tag{1.2}$$

$$\Gamma_{ijk} = m\left[\left(\frac{\partial\alpha_i}{\partial q_j} - \frac{\partial\alpha_j}{\partial q_i}\right)\alpha_k + \left(\frac{\partial\beta_i}{\partial q_j} - \frac{\partial\beta_j}{\partial q_i}\right)\beta_k\right] = -\Gamma_{jik}$$

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$$L^* = \frac{1}{2} J_{22}\theta'^2 + \frac{1}{2}(J_{11}\sin^2\theta + J_{33}\cos^2\theta - 2J_{13}\sin\theta\cos\theta)\varphi'^2 +$$
$$\frac{1}{2} J_{33}\psi'^2 - (J_{12}\sin\theta + J_{23}\cos\theta)\theta'\varphi' - J_{23}\theta'\psi' +$$
$$(J_{33}\cos\theta - J_{13}\sin\theta)\varphi'\psi' + mg(\chi_1\sin\theta + \zeta\cos\theta)$$
$$J_{11} = (A\sin^2\varphi + B\cos^2\varphi)\cos^2\theta + C\sin^2\theta + m[\chi_2^2 + (\chi_1\sin\theta + \zeta\cos\theta)^2]$$
$$J_{22} = (A\cos^2\varphi + B\sin^2\varphi) + m(\chi_1^2 + \zeta^2)$$
$$J_{33} = (A\sin^2\varphi + B\cos^2\varphi)\sin^2\theta + C\cos^2\theta + m[\chi_2^2 + (\chi_1\cos\theta - \zeta\sin\theta)^2]$$
$$J_{12} = (A - B)\sin\varphi\cos\varphi\cos\theta - m\chi_2(\chi_1\cos\theta - \zeta\sin\theta)$$
$$J_{13} = (A\sin^2\varphi + B\cos^2\varphi - C)\sin\theta\cos\theta - m(\chi_1\cos\theta - \zeta\sin\theta)(\chi_1\sin\theta + \zeta\cos\theta)$$
$$J_{23} = -(A - B)\sin\varphi\cos\varphi\sin\theta + m\chi_2(\chi_1\sin\theta + \zeta\cos\theta)$$

where J_{ij} are, respectively, the axial $(i = j)$ and centrifugal $(i \neq j)$ moments of inertia of the body relative to the axes of the system of coordinates $Kx_1y_1z_1$, whose origin is at the contact point of the body and supporting plane, axis Kz_1 directed vertically upward, axis Ky_1 parallel to the line of nodes and pointing in the direction from which the turning of the vertical line passing through the body center of mass by angle θ to congruity with axis $G\zeta$, is counterclockwise, and axis Kx_1 is normal to the plane Ky_1z_1, thus forming a right-hand coordinate system.

2. Function L^* is, obviously, independent of ψ; taking into account that $\beta = -\partial\alpha/\partial\psi$, $\alpha = \partial\beta/\partial\psi$, we conclude that the coefficients Γ_{ijk} in Eqs.(1.2) also have that property. Hence ψ is an ignorable coordinate in the sense used in /1,2/, and the input system can perform steady motions of the form

$$\theta = \theta_0,\ \theta' = 0,\ \varphi = \varphi_0,\ \varphi' = 0,\ \psi' = \psi_0' \equiv \omega \tag{2.1}$$

Under these conditions the body contacts the horizontal plane at one and the same of its points and rotates about the vertical line passing through the point, while its center of mass describes a circle which is parallel to the supporting plane and whose center is on the body rotation axis.

The three constants $\theta_0, \varphi_0, \omega$ in (2.1) satisfy the system of two equations

$$(\xi\sin\varphi + \eta\cos\varphi)\cos\theta - \zeta\sin\theta = -\frac{J_{13}\omega^2}{mg} \tag{2.2}$$
$$\xi\cos\varphi - \eta\sin\varphi = \frac{J_{23}\omega^2}{mg}$$

which is to be used for determining θ_0 and φ_0, considering ω as an arbitrary constant. Eliminating ω^2 we obtain

$$(B - C)\,\xi\sin\theta\cos\varphi\cos\theta + (C - A)\,\eta\sin\theta\sin\varphi\cos\theta + (A - B)\,\zeta\sin^2\theta\sin\varphi\cos\varphi = 0 \tag{2.3}$$

which under conditions of steady motion (2.1) must be satisfied by θ and φ or, what is the same, the directional cosines $\gamma_1 = \sin\theta\sin\varphi$, $\gamma_2 = \sin\theta\cos\varphi$, $\gamma_3 = \cos\theta$ of possible axes of permanent rotations of a heavy solid body on an absolutely rough horizontal plane.

Among the kinematically possible permanent rotation axes whose directional cosines satisfy Eq.(2.3) not all are dynamically possible, but only those for which the inequality $\omega^2 \geqslant 0$ follows from Eqs.(2.2). This leads to the condition

$$(A - B)\sin\theta\sin\varphi\cos\varphi(\xi\cos\varphi - \eta\sin\varphi) \leqslant m(\xi\cos\varphi - \eta\sin\varphi)^2[(\xi\sin\varphi + \eta\cos\varphi)\sin\theta + \zeta\cos\theta] \tag{2.4}$$

whose right-hand side is nonpositive, since in it in the expressions in brackets the height is taken with the minus sign of the body center of mass above the supporting plane.

This implies that, although the equations which must be satisfied by the directional cosines of possible permanent rotation axes in the problems of motion of a heavy solid body on an absolutely smooth and absolutely rough planes, the domain of dynamically admissible axes is wider in the first case (in the case of absolutely smooth plane the equations that corresponds to (2.3) is of the same form, and the condition corresponding to (2.4) is obtained from the latter by equating its right side to zero /3/).

Note also certain differences in the problems of permanent rotations of a heavy solid body on an absolutely smooth and on an absolutely rough planes. In the first case the body rotates about the vertical line passing through the body center of mass, with the center of mass stationary, while the point of its contact with the supporting plane describes on the latter a circle (sliding on it). In the second case the body rotates about the vertical line passing through the point of contact of the body and the supporting plane, with the body center of mass describing a circle parallel to the supporting plane.

3. Consider the stability of permanent rotations of the input system. The characteristic equation that corresponds to the system of equations of perturbed motion is of the form

$$\begin{aligned}
&\lambda (a_0\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4) = 0 \\
&a_0 = \{K_{11}K_{22} - K_{12}^2 - K_{33}^{-1} (K_{11}K_{23}^2 + K_{22}K_{13}^2 + 2K_{12}K_{13}K_{23})\}^0 > 0 \\
&a_1 = \omega \{- mh (r_2 - r_1) (K_{12} + K_{33}^{-1} K_{13}K_{23}) + \\
&\quad K_{33}^{-1}\omega^2 g^{-1} [K_{13}K_{23}(K_{11}r_1 - K_{22}r_2) + K_{12} (K_{13}^2 r_1 - K_{23}^2 r_2)]\}^\circ \\
&a_2 = mg [(K_{22} - K_{33}^{-1}K_{23}^2) (r_2 - h) + (K_{11} - K_{33}^{-1}K_{13}^2) r_1 -)]^\circ + \\
&\quad \omega^2 \{m^2h^2 r_1 r_2 + mh [(K_{33} - K_{11}) r_2 + (K_{33} - K_{22}) r_1 + \\
&\quad K_{33}^{-1} (K_{13}^2 (r_1 + 2r_2) + K_{23}^2 (2r_1 + r_2))] + (K_{33} - K_{11}) \times \\
&\quad (K_{33} - K_{22}) + K_{11}K_{22} - 2K_{12}^2 + 3 (K_{13}^2 + K_{23}^2) + K_{33}^{-1} \times \\
&\quad [K_{11} (K_{13}^2 - 4K_{23}^2) + K_{22} (K_{23}^2 - 4K_{13}^2) - \\
&\quad 10 K_{33}^{-1} K_{12}K_{13}K_{23}]\}^\circ + K_{33}^{-1}\omega^4 g^{-1} \{mhr_1r_2 (K_{13}^2 + K_{23}^2) + \\
&\quad (K_{33} + K_{22} - K_{11}) K_{23}^2 r_2 + (K_{33} - K_{22} + K_{11}) K_{13}^2 r_1 - 2K_{12}K_{13}K_{23} (r_1 + r_2)\}^\circ \\
&a_3 = \omega^3 \{mh (r_2 - r_1) (K_{12} + 3K_{33}^{-1}K_{13}K_{23}) + K^{-1}{}_{33}\, \omega^2 g^{-1} \times \\
&\quad [K_{13}K_{23} (K_{11}r_1 - K_{22}r_2) + K_{12} (K_{13}^2 r_1 - K_{23}^2 r_2) + \\
&\quad K_{13}K_{23} (K_{33} - K_{11} - K_{22}) (r_2 - r_1)]\}^\circ \\
&a_4 = [m^2 g^2 (r_1 - h) (r_2 - h)]^\circ + mg\omega^2 \{K_{33} (r_1 + r_2 - 2h) - \\
&\quad K_{11} (r_2 - h) - K_{22} (r_1 - h) + mh [2r_1r_2 - h (r_1 + r_2)] + \\
&\quad 4K_{33}^{-1}[K_{13}^2 (r_2 - h) + K_{23}^2 (r_1 - h)]\}^\circ + \\
&\quad \omega^4 \{m^2h^2 r_1 r_2 + mh [(K_{33} - K_{11}) r_2 + (K_{33} - K_{22}) r_1] + \\
&\quad 2mhK_{33}^{-1} [2 (K_{13}^2 r_2 + K_{23}^2 r_1) - (K_{13}^2 r_1 + K_{23}^2 r_2)] + \\
&\quad 2mr_1r_2K_{33}^{-1}(K_{13}^2 + K^2{}_{23}) + (K_{33} - K_{11}) (K_{33} - K_{22}) - \\
&\quad K_{12}^2 - 8K_{33}^{-1} K_{12}K_{13}K_{23} + 4K_{33}^{-1}[K_{23}^2 (K_{33} - K_{11}) + \\
&\quad K_{13}^2 (K_{33} - K_{22})]\}^\circ + 2K_{33}^{-1}\omega^6 g^{-1} \{(K_{13}^2 + K_{23}^2)\, mhr_1r_2 + \\
&\quad (K_{33} - K_{11}) K_{23}^2 r_2 + (K_{33} - K_{22})\, K_{13}^2 r_1 - K_{12}K_{13}K_{23} (r_1 + r_2)\}^\circ \\
&K_{11} = J_{11} \cos^2\alpha + J_{22} \sin^2\alpha - 2J_{12} \sin\alpha\cos\alpha, \quad K_{13} = J_{13}\cos\alpha + J_{23}\sin\alpha \\
&K_{22} = J_{11} \sin^2\alpha + J_{22}\cos^2\alpha + 2J_{12}\sin\alpha\cos\alpha, \quad K_{23} = - J_{13}\sin\alpha + J_{23}\cos\alpha \\
&K_{12} = (J_{11} - J_{22}) \sin\alpha\cos\alpha + J_{12} (\cos^2\alpha - \sin^2\alpha), \; K_{33} = J_{33}
\end{aligned} \tag{3.1}$$

where K_{ij} are, respectively, the axial $(i = j)$ and centrifugal $(i \neq j)$ moments of inertia of the body relative to the axes of the coordinate system $Kx_2y_2z_1$ with its origin at the point of contact of the body and supporting plane, axes Kx_2 and Ky_2 of that system coincide with the directions of the principal radii of curvature $r_1 = r_1(\theta, \varphi)$ and $r_2 = r_2(\theta, \varphi)$, respectively, at point K; $h = h(\theta, \varphi)$ is the distance of the center of mass of the body over the support plane; $\alpha = \alpha(\theta, \varphi)$ is the angle between axes Kx_1 and Kx_2 measured from axis Kx_1 towards axis Ky_1; the superscript zero indicates that the respective function of variables θ and φ is calculated for $\theta = \theta_0$, $\varphi = \varphi_0$.

Equation (3.1) has always a single zero root. If at least one root of Eq.(3.1) lies in the right half-plane, solution (2.1) is unstable. If, however, all roots of equation

$$a_0\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0 \tag{3.2}$$

are in the left half-plane, then we have a particular case of the critical case of a single zero root, and solution (2.1) is stable. Although the input system is conservative, the system is asymptotically stable with respect to a part of variables which define deviations of θ, $\theta^{\cdot}$, φ, $\varphi^{\cdot}$, $\psi^{\cdot}$ from their values on the manifold of steady motions.

All roots of Eqs.(3.2) lie in the left half-plane then and only then when conditions

$$a_1 > 0,\ a_2 > 0,\ a_4 > 0,\ a_1a_2a_3 - a_1^2a_4 - a_0a_3^2 > 0 \tag{3.3}$$

are satisfied. When the sign of at least one of inequalities (3.3) is reversed, Eq.(3.1) has a root in right half-plane.

4. Thus conditions (3.3) are the necessary (accurate to the inequality sign) and sufficient conditions of stability of permanent rotations of a heavy solid body on an absolutely rough plane.

Note that, since θ_0 and φ_0 depend on ω^2 (see(2.2)), the coefficients a_0, a_2, a_4 of Eq.(3.2) are even and a_1 and a_3 odd functions of ω. Hence the last three conditions in (3.3) impose constraints on the distribution of mass, the surface geometry and the magnitude of the body angular velocity, while the first of them does so on the sign of angular velocity. This means that under otherwise equal conditions, stability of permanent rotations of a heavy solid body on an absolutely rough horizontal plane depends on the sense of rotation: motion is stable in one direction and unstable in the opposite direction.

This dependence and, also, the possibility of asymptotic stability with respect to a part of variables in the absence of external dissipative forces makes the considered here problem substantially different from that of stability of permanent rotations of a heavy solid body on an absolutely smooth horizontal plane /3/. In the latter, stability conditions are independent of the sense of rotation of the body, while in the absence of external dissipative forces an asymptotic stability is not possible with respect to any of the variables.

When the body rotates about one of its principal axes of its ellipsoid of inertia calculated relative to the contact point of the body and the support plane (when $K_{13} = K_{23} = 0$), conditions (3.3) are considerably simplified, and assume the form

$$a_1^* > 0,\ a_4^* > 0,\ (a_2^* - a_0^*\omega^2)\,\omega^2 - a_4^* > 0 \tag{4.1}$$

where the asterisk means that in the respective coefficents of Eq.(3.1) K_{13} and K_{23} are assumed equal zero, as indicated in /2,4,5/.

Note that, if the rotation of a body about one of the principal axes of its ellipsoid of inertia relative to point K is to be asymptotically stable with respect to a part of variables and the stability conditions are to depend on the sense of rotation, the principal radii of curvature of the body surface at the point of its contact with the support plane must not be equal to each other ($r_1 \neq r_2$) and their directions must not be coincide with the two other principal axes of the body ellipsoid of inertia relative to that point ($K_{12} \neq 0$). These conditions do not apply in the case of rotation about any arbitrary axis ($K_{12}(r_2 - r_1)$ can be zero).

When $\omega = 0$ Eqs.(2.2) determine the equilibrium position of a heavy solid body on an absolutely rough horizontal plane, and imply that in its equilibrium position the body center of mass is on the vertical line passing through its point of contact with the support plane. The investigation of properties of the body potential energy extremum in conformity with /6/ shows that the equilibrium is stable (with respect to $\theta, \theta^{\cdot}, \varphi, \varphi^{\cdot}, \psi^{\cdot}$) if the body center of mass is below both centers of curvature of the body surface at its point of contact with the support plane; otherwise it is unstable /5/. Similar statements hold also in the case of the absolutely smooth plane /3,5/. Thus the results of investigations of the equilibrium positions of a heavy solid body on an absolutely smooth and absolutely rough horizontal plane are to some extent equivalent, which is not the case with the results of investigation of permanent rotation stability.

5. Let us now fix the stationary point of contact of a heavy solid body with an absolutely rough horizontal surface in the case of permanent rotations. The investigated nonholonomic system then becomes holonomic: the system of heavy solid body with a fixed point. We shall use the introduced above variables and notation, unconventional in problems of motion of a heavy solid body with a fixed point, where the origin of coordinate system rigidly attached to the body is usually at the fixed point (not to the center of mass) with its axes directed along the principal axes of the body ellipsoid of inertia relative to the fixed point (not to the center of mass). It can, then, be shown that the equations of motion of the heavy solid body with a fixed point are obtained from Eqs.(1.2) with $\Gamma_{ijk} = 0$ under the condition that ξ, η, ζ (the coordinates of the fixed point in the system $G\xi\eta\zeta$) in expressions for L^* and J_{ij} are constant.

It appears that the equation of the permanent rotation manifold of a heavy solid body with a fixed point is also of the form (2.2). This explains why in the problems considered here the domains of kinematically possible (2.3) and dynamically admissible (2.4) axes of permanent rotations coincide (within the indicated above meaning of ξ, η, ζ).

However, in spite of the complete geometric analogy, problems of permanent rotations of a heavy solid body with a fixed point and on an absolutely rough horizontal plane substantially differ. In the first case /7/, as well as in that of a body on an absolutely smooth plane, the conditions of stability are independent of the sense of rotation of the body and in the absence of external dissipative forces no asymptotic stability is possible with respect to any of the variables.

Note that the respective characteristic equation for a body with fixed point is obtained from Eq.(3.1) with $r_1 = r_2 = 0$, i.e. when $a_1 = a_3 = 0$, $a_2 = a_{20}$, $a_4 = a_{40}$, where the subscript zero indicates that in the respective expression $r_1 = r_2 = 0$ must be set; h is the height of the body center of mass above the horizontal plane passing through the fixed point.

Thus the possibility of rocking plays an important part in the unusual behavior of the heavy solid body on an absolutely rough horizontal plane in perturbed motions (i.e. variation of ξ, η, ζ, whose first derivatives with respect to θ and φ linearly depend on the principal radii of curvature of the body surface at the point of its contact with the support plane).

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